RIESZ - LEMMA

Theorem (A):- Let M be a closed proper subspace of a normed linear space N, and let a be a real number such that $0 < a < 1$. Then there exists a vector $x_0 \in N$ such that $\|x_a\| = 1$ and $\|x - x_0\| \geq a$ for all $x \in M$.

Proof:- Any $x_1 \in N - M$ and let

$$h = \inf_{x \in M} \|x - x_1\| = d(x_1, M).$$

it is clear that h must be strictly greater than zero, for otherwise we would have

$$h = 0 \Rightarrow d(x_1, M) = 0 \Rightarrow x_1 \in M \quad (\because M \text{ is closed})$$

which contradicts the way in which x_1 was chosen. Since $0 < a < 1$, we have $\frac{h}{a} > h$.

Hence by the definition of infimum, there exists $x_0 \in M$ such that

$$h < \|x_0 - x_1\| \leq a^{-1} h \quad \text{——— (1)}$$

Let $x_0 = k(x_1 - x_0)$ where $k = \|x_1 - x_0\| > 0$

then $\|x_a\| = k\|x_1 - x_0\| = kk^{-1} = 1$.

Now let $x \in M$ be arbitrary. Then $k^{-1}x + x_0 \in M$ also and so

$$\|x - x_a\| = \|x - k(x_1 - x_0)\|$$

$$= k\|(k^{-1}x + x_0) - x_1\| \geq kh \quad \text{— (2)}$$

$\because h = \inf_{x \in M} \|x - x_1\|$ and $k^{-1}x + x_0 \in M$, we have

$$\|(k^{-1}x + x_0) - x_1\| \geq h$$

But $kh = \|x_1 - x_0\|^{-1} h \geq a$ by (1) ——— (3)

From equation (2) and (3), we have

$$\|x - x_0\| \geq a \text{ for all } x \in M.$$

Theorem (B): — Let N be a normed linear space and suppose the set $S = (x \in N : \|x\| = 1)$ is compact. Then N is finite dimensional.

Proof: — We know that in a metric space, a subset is compact iff it is sequentially compact i.e. iff every sequence has a convergent subsequence. Since S is given to be compact, every sequence in S must have a convergent subsequence.

Suppose if possible, N is not finite dimensional. Choose $x_1 \in S$ and let N_1 be the subspace spanned by x_1. Then N_1 is a proper subspace of N since N_1 is finite dimensional and therefore it is closed.

Hence by Reisz-lemma there exists a vector $x_2 \in S$ such that $\|x_2 - x_1\| \geq \frac{1}{2}$.

Let N_2 be the closed proper subspace of N generated by x_1, x_2, then as before there must exist $x_3 \in S$ such that

$$\|x_3 - x\| \geq \frac{1}{2} \text{ if } x \in N.$$

Proceeding inductively, we obtain an infinite sequence $\langle x_n \rangle$ of vectors in S such that $\|x_n - x_m\| \geq \frac{1}{2}$.

This sequence can therefore have no convergent subsequence. But this contradicts the hypothesis that S is compact. Hence N must be finite-dimensional.

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